Discrete Mathematics 13 (1975) 383–390.

ON THE RATIO OF OPTIMAL INTEGRAL AND FRACTIONAL COVERS

L. LOVÁSZ
Eötvös L. University, Budapest, Hungary

Received 20 January 1975

It is shown that the ratio of optimal integral and fractional covers of a hypergraph does not exceed $1 + \log d$, where d is the maximum degree. This theorem may replace probabilistic methods in certain circumstances. Several applications are shown.

A *hypergraph* is a finite collection of non-empty finite sets called *edges*. The elements of edges are called vertices. The set of vertices of a hypergraph H is denoted by $V(H)$.

The *degree* of $x \in V(H)$ is the number of edges containing x. $d(H)$ denotes the maximum degree in H. In $d(H)$ and other functions to be defined we remove the argument H if no confusion can arise.

A *k-matching* of H is a collection M of edges (the same edge may occur more than once) such that each point belongs to at most k of them. The maximum number of edges in a k-matching is denoted by $\nu_k(H)$. Thus $\nu(H) = \nu_1(H)$ is the maximum number of disjoint edges.

A k-matching is *simple* if no edge occurs in it more than once. If $\tilde{\nu}_k$ is the maximum number of edges in simple k-matchings, then
$\tilde{\nu}_k \leq \nu_k$.

A *k-cover* of H is a collection T of points such that any edge contains at least k points of T. The minimum number of points in a k-cover will be denoted by $\tau_k(H)$. Thus $\tau(H) = \tau_1(H)$ is the minimum number of representing points.

A *fractional matching* is a system of weights $\{\omega_E : E \in H\}$ such that

$$\omega_E \geq 0, \qquad \sum_{x \in E} \omega_E \leq 1$$

for every point x; a *fractional cover* is a system of weights $\{t_x : x \in V(H)\}$ such that

$$\sum_{x \in E} t_x \geq 1$$

for every edge E. We define

$$\nu^*(H) = \min \sum_E \omega_E, \qquad \tau^*(H) = \max \sum_x t_x,$$

where the extrema are taken over all fractional matchings resp. covers. By the duality theorem, we have $\nu^* = \tau^*$. It is easy to see that

$$\nu \leqslant \nu_k/k \leqslant \nu^* = \tau^* \leqslant \tau_k/k \leqslant \tau.$$

In some cases equalities hold here; such are known minimax theorems as König's theorem, Menger's theorem on graphs, etc. (e.g., [2, Ch. 20, pp 448–475] and [3, 5, 7–9]). But even if equality does not hold it is of interest to obtain inequalities going in the opposite way; several results in combinatorics can be formulated this way, e.g., Vizing's theorem on chromatic index [13], Erdös' and Pósa's theorem on representing all circuits of a graph by a small number of points [4], results of Gallai [5], etc.

In this note we give a general inequality and some applications.

If we want to obtain a small number of points which represent all edges, we may follow this procedure. Let x_1 be a point with maximum degree; supposing $x_1, ..., x_i$ are already selected and they still do not cover all edges. Let x_{i+1} be a point which covers the most number of new edges. If $x_1, ..., x_i$ cover all edges, we stop. This way we obtain a set $\{x_1, ..., x_t\}$ of points covering all edges. This procedure is called the *greedy cover algorithm*. The number t of covering points is by no means always equal to $\tau(H)$ and is not uniquely defined by H.

Theorem 1. *If H is a hypergraph and any greedy cover algorithm produces t covering points, then*

$$t \leqslant \frac{\tilde{\nu}_1}{1\cdot 2} + \frac{\tilde{\nu}_2}{2\cdot 3} + \ldots + \frac{\tilde{\nu}_{d-1}}{(d-1)\cdot d} + \frac{\tilde{\nu}_d}{d}$$

(clearly $\tilde{\nu}_d = |E(H)|$).

Corollary 2. $\tau \leqslant (1 + \frac{1}{2} + \ldots + 1/d)\tau^* < (1 + \log d)\tau^*$.

Proof of Theorem 1. Let t_i denote the number of steps in the greedy cover algorithm in which the chosen point covers i new edges. After $t_d + \ldots + t_{i+1}$ steps, the hypergraph H_i formed by the uncovered edges has degree $\leqslant i$, hence

$$|E(H_i)| \leqslant \tilde{\nu}_i.$$

On the other hand,

$$|E(H_i)| = it_i + \ldots + 2t_2 + t_1 .$$

Thus we have

$$it_i + \ldots + 2t_2 + t_1 \leqslant \tilde{\nu}_i \qquad (i = 1, \ldots, d).$$

Multiply these inequalities in order by $1/1 \cdot 2$, $1/2 \cdot 3$, ..., $1/(d-1) \cdot d$, and sum, then we obtain on the left-hand side a given t_i with coefficient

$$i\left(\frac{1}{i(i+1)} + \ldots + \frac{1}{d(d-1)} + \frac{1}{d}\right) =$$

$$= i\left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+1} - \frac{1}{i+2} + \ldots + \frac{1}{d-1} - \frac{1}{d} + \frac{1}{d}\right) = 1 ;$$

on the right-hand side we get

$$\sum_{i=1}^{d-1} \frac{\tilde{\nu}_i(H)}{i(i+1)} + \frac{\tilde{\nu}_d(H)}{d} .$$

Since $t = t_1 + \ldots + t_d$, this yields

$$t \leqslant \frac{\tilde{\nu}_1}{1 \cdot 2} + \frac{\tilde{\nu}_2}{2 \cdot 3} + \ldots + \frac{\tilde{\nu}_{d-1}}{(d-1) \cdot d} + \frac{\tilde{\nu}_d}{d}$$

as stated.

Proof of Corollary 2. We have $\tau \leqslant t$, $\tilde{\nu}_i \leqslant \nu_i \leqslant i\nu^* = i\tau^*$; setting in these estimates, the inequality follows.

First we deduce a combinatorial application. A hypergraph H is said to be 2-chromatic if its points can be 2-colored such that no edge is monochromatic. H is *r-uniform* if each edge has exactly r elements. It is known [1, 4] that an r-uniform, non-2-chromatic hypergraph must have $\geqslant c \cdot 2^r \log r$ edges. On the other hand, we have the following:

Theorem 3 (Erdös [4]). *There is a non-2-chromatic r-uniform hypergraph with not more than $c \cdot 2^r \cdot r^2$ edges.*

Erdös gave a probabilistic proof but he pointed out that a certain "successive optimal choice" method (like the greedy cover algorithm) also yields the result. So this proof here is not really new except that it uses the more convenient language of covers.

Proof. Given a set X of $2N > r$ points, form a hypergraph H as follows. Let $V(H)$ consist of all r-tuples formed by elements of X. Moreover, for every partition $P = \{X_1, X_2\}$ of X, we form the set $\bar{P}$ of all r-subsets of X which entirely belong to X_1 or X_2. Now if T is a collection of r-tuples, then T is a 1-cover of H iff T is non-2-chromatic. Thus, there exists a non-2-chromatic r-uniform hypergraph with $\tau(H)$ edges.

By Corollary 2 $\tau(H) \leqslant (1 + \log d(H))\tau^*(H)$. If we set

$$t_x = t = 1/2\binom{N}{r} \qquad (x \in V(H)),$$

we obtain a fractional cover; for if E is any edge of H, i.e. E is the set of r-tuples contained in X_1 or X_2 where $\{X_1, X_2\}$ is a partition of X, then

$$|E| = \binom{|X_1|}{r} + \binom{|X_2|}{r} \geqslant 2\binom{N}{r}.$$

Hence

$$\sum_{x \in E} t_x = |E|, \qquad t \geqslant 1.$$

Thus,

$$\tau^*(H) \leqslant \sum_x t_x = \binom{2N}{r} t = \binom{2N}{r}/2\binom{N}{r}.$$

Also,

$$d = 2^{2N-r} < e^{2N-r}$$

and thus

$$\tau(H) \leqslant (1 + 2N - r)\binom{2N}{r}/2\binom{N}{r}.$$

Choosing $N = n^2$ we will have

$$\binom{2N}{r}/2\binom{N}{r} < c \cdot 2^r$$

and thus

$$\tau(H) < c \cdot r^2 \cdot 2^r.$$

The following theorem is a slight generalization of a theorem of Lorentz [6].

Theorem 4. *Let G be a group and $A \subseteq G$. Set $|A| = r$, $|G| = n$. Then there exists a set $B \subseteq G$ such that $AB = G$ and*

$$|B| \leqslant \frac{1 + \log r}{r} n.$$

Proof. Consider the sets $A^{-1}g$, $g \in G$. They form a hypergraph H, in which each degree is r and also each edge is an r-tuple. Therefore $t_x = 1/r$ is a fractional cover and hence $\tau^*(H) \leq n/r$ (it is quite easy to see that one must have equality here). Also, $d(H) = r$. Thus, by Corollary 2,

$$\tau(H) \leq (1 + \log r)n/r.$$

So let $B \subseteq G$ be a cover of H. Then for each $g \in G$,

$$B \cap A^{-1}g \neq \emptyset,$$

i.e., there are $b \in B$, $a \in A$ such that

$$b = a^{-1}g, \qquad g = ab.$$

Thus $AB = G$.

Other results on complementary sequences would follow similarly.

Our next application has an information theoretical background but it can be formulated purely combinatorially. The following theorem is due to McEliece and Posner [10] (see this reference for information theoretical background).

Define, for two hypergraphs H_1, H_2, their *product* $H_1 \times H_2$ by

$$H_1 \times H_2 = \{E \times F: E \in H_1, F \in H_2\},$$

and set

$$H^k = \underbrace{H \times \ldots \times H}_{k \text{ times}}.$$

We remark that $V(H_1 \times H_2) = V(H_1) \times V(H_2)$.

Theorem 5. *Let H be any hypergraph. Then*

$$\lim_{k \to \infty} \sqrt[k]{\tau(H^k)} = \tau^*(H).$$

Proof. Observe that

$$\nu^*(H_1 \times H_2) \geq \nu^*(H_1)\, \nu^*(H_2),$$

in fact, if $\omega'_E (E \in H_1)$, $\omega''_E (E \in H_2)$ are optimal fractional matchings of H_1 and H_2, respectively, then

$$\omega_{E_1 \times E_2} = \omega_{E_1} \omega_{E_2}$$

is a fractional matching in $H_1 \times H_2$ and

$$\sum_{E_1,E_2} \omega_{E_1 \times E_2} = \sum_{E_1} \omega'_{E_1} \sum_{E_2} \omega''_{E_2} = \nu^*(H_1)\, \nu^*(H_2).$$

A similar argument shows that

$$\tau^*(H_1 \times H_2) \leqslant \tau^*(H_1)\, \tau^*(H_2)$$

and thus by $\nu^* = \tau^*$ it follows that

$$\tau^*(H_1 \times H_2) = \tau^*(H_1)\, \tau^*(H_2).$$

Hence

$$\sqrt[k]{\tau(H^k)} \geqslant \sqrt[k]{\tau^*(H^k)} = \tau^*(H).$$

What is left is to estimate $\tau(H^k)$ from the above. It is easy to see that $d(H^k) = d(H)^k$ and hence by Corollary 2,

$$\tau(H^k) \leqslant (1 + k \log d(H))\, \tau^*(H)^k.$$

Thus

$$\sqrt[k]{\tau(H^k)} \leqslant (\sqrt[k]{1 + k \log d(H)})\tau^*(H).$$

Since

$$\sqrt[k]{1 + k \log d(H)} \to 1 \qquad (k \to \infty),$$

this proves the assertion.

We give one more application, which is believed to be new. Let G be a graph. The *k-tuple chromatic number* [1] of G is the least number l such that it is possible to associate a k-subset $C(x)$ of $\{1, ..., l\}$ with every point x of G in such a way that $C(x) \cap C(y) = \emptyset$ for $(x, y) \in E(G)$. We denote the k-tuple chromatic number of G by $\chi_k(G)$. Clearly $\chi(G) = \chi_1(G)$ is the ordinary chromatic number.

Let us define a hypergraph I_G as follows: For every $x \in V(G)$, denote by E_x the set of all maximal independent sets in G containing x and let

$$I_G = \{E_x : x \in V(G)\}.$$

Now observe that $\tau(I_G) = \chi_k(G)$ for the left-hand side is the minimum number of independent sets which cover every point at least k times; and as noted in [12], the same interpretation can be given for $\chi_k(G)$.

[1] Cf. Rosenfeld [11], Stahl [12].

Hence we can estimate $\chi_k(G)$ as follows, using Corollary 2:

$$\chi_k(G) = \tau_k(I_G) \geqslant k \cdot \tau^*(I_G) \geqslant \frac{k}{1 + \log d(I_G)} \tau(I_G)$$

$$= \frac{k}{1 + \log d(I_G)} \chi(G);$$

now if $u \in V(I_G)$, then u is an independent set in G and the degree of u is $|u|$. Hence

$$d(I_G) = \alpha(G);$$

the maximum number of independent points in G. Hence we have the following:

Theorem 6.

$$\chi_k(G) \geqslant \frac{k}{1 + \log \alpha(G)} \chi(G).$$

Using Theorem 1 instead of Corollary 2, we obtain a result of simpler form. Let $\omega_k(G)$ denote the maximum number of points in G which span no independent $(k+1)$-set. Thus $\omega_1(G)$ is the number of points in a maximum clique. Then

$$\omega_k(G) = \tilde{\nu}_k(I_G)$$

and thus

$$\chi(G) \leqslant \sum_{k=1}^{\alpha(G)-1} \frac{\omega_k(G)}{k(k+1)} + \frac{|V(G)|}{\alpha(G)} .$$

If $X \subseteq V(G)$ spans no independent $(k+1)$-set, then denoting by G' the subgraph induced by X we have

$$\frac{\omega_k(G)}{k} = \frac{|V(G')|}{\alpha(G')} \leqslant \max_H \frac{|V(H)|}{\alpha(H)} ,$$

where H ranges over all induced subgraphs of G. Thus we obtain the following:

Theorem 7. *For any graph G,*

$$\chi(G) \leqslant (1 + \log \alpha(G)) \max_H \frac{|V(H)|}{\alpha(H)} ,$$

where H ranges over all induced subgraphs of G.

References

[1] J. Peck, oral communication.
[2] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).
[3] J. Edmonds, Sub modular functions, matroids and certain polyhedra, in: Combinatorial Structures and its Applications (Gordon and Breach, New York, 1970) 69–87.
[4] P. Erdős, On a combinatorial problem II, Acta Math. Acad. Sci. Hungar. 15 (1964) 445–447.
[5] T. Gallai, Maximum-minimum Sätze und verallgemeinerte Faktoren von Graphen, Acta Math. Acad. Sci. Hungar. 12 (1961) 131–173.
[6] G. Lorentz, On a problem in additive number theory, Proc. Am. Math. Soc. 5 (1954) 838–841.
[7] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (1972) 253–267.
[8] L. Lovász, 2-matchings and 2-covers of hypergraphs, to appear.
[9] C. Luchesi and D.H. Younger, A minimax theorem for directed graphs, to appear.
[10] R.J. McEliece and E.C. Posner, Hide and seek, data storage, and entropy, Am. Statist. 42 (1971) 1706–1716.
[11] M. Rosenfeld, On a problem of C.E. Shannon in graph theory, Proc. Am. Math. Soc. 18 (1967) 315–319.
[12] S. Stahl, n-tuple colorings and associated graphs, J. Combin. Theory, to appear.
[13] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz. 3 (1964) 25–30 (in Russian); see also [2].